Topological Models for (Full) Belief and Belief Revision

Aybüke Özgün

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December 2, 2013

Joint work with Alexandru Baltag, Nick Bezhanishvili and Sonja Smets.

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Topological Models for (Full) Belief and Belief Revision

Some History and Motivation	Stalnaker's Logic KB	Our Work 00000 00 0000	Future Work

A *topological space* is a pair (X, τ) where X is a non-empty set and $\tau \subseteq \mathcal{P}(X)$ such that

- $X, \emptyset \in \tau$
- τ is closed under finite intersection and arbitrary union.

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An open set containing $x \in X$ is called *open neighborhood* of x.

• Interior operator:

 $\begin{array}{rcl} \mathrm{Int}:\mathcal{P}(X) & \to & \mathcal{P}(X) \\ & \mathrm{Int}(A) & \mapsto & \text{the largest open set contained in } A \end{array}$

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• Interior operator:

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• Closure operator:

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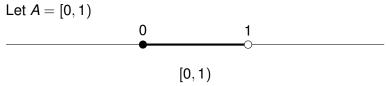
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•
$$\operatorname{Cl}(A) = X \setminus \operatorname{Int}(X \setminus A)$$

Some History and Motivation	Stalnaker's Logic KB	Our Work 00000 00 0000	Future Work

Example:

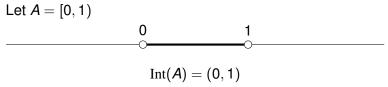
Consider the real line and the topology of open intervals and their countable unions on \mathbb{R} .



Some History and Motivation	Stalnaker's Logic KB	Our Work 00000 00 0000	Future Work

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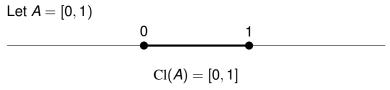
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Stalnaker's Logic KB

Our Work 00000 00 0000 Future Work

Topological Semantics for Knowledge

Topological Models for (Full) Belief and Belief Revision

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Topological Semantics for Knowledge

 $(\mathcal{L}_{\mathcal{K}}) \varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi \mid \mathcal{K}\varphi$

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A topological model is a tuple $\mathcal{M} = (X, \tau, \nu)$ where X and τ as before and ν : Prop $\rightarrow \mathcal{P}(X)$ is valuation function.

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We can think of

- open sets $U \in \tau$ as pieces of evidence, and
- open neighborhoods of the actual world as pieces of truthful evidence.

Some History and Motivation	Stalnaker's Logic KB	Our Work 00000 00 0000	Future Work

Theorem (McKinsey and Tarski, 1944)

S4 is sound and complete wrt the class of all topological spaces.

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positively introspective

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positively introspective

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but not necessarily negatively introspective

$$\neg K\varphi \rightarrow K\neg K\varphi.$$

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Topological Models for (Full) Belief and Belief Revision

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Downsides:

- it entails *the necessity of error*: there is at least one false belief in all worlds of every topological model.
- 2 it can easily be "Gettierized":

$$\mathbf{K}\varphi := \mathbf{B}\varphi \wedge \varphi$$

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Our Focus

Given the interior-based topological semantics for knowledge:

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Given the interior-based topological semantics for knowledge:

how can we construct a topological semantics for belief which can also address the problem of understanding the relation between knowledge and belief?

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Our Focus

Given the interior-based topological semantics for knowledge:

- how can we construct a topological semantics for belief which can also address the problem of understanding the relation between knowledge and belief?
- A how to extend this setting to static and dynamic belief revision?

Stalnaker's Logic KB

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Stalnaker's Logic KB

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Epistemic-Doxastic Axioms	
$\boxed{ K(\varphi \to \psi) \to (K\varphi \to K\psi) }$	Knowledge is additive
Karphi ightarrow arphi	Knowledge implies truth
Karphi ightarrow KKarphi	Positive introspection for K
$oldsymbol{B}arphi ightarrow eg B abla arphi ightarrow eg B abla arphi$	Consistency of belief
$egin{array}{c} egin{array}{c} eta & eb$	(Strong) positive introspection of <i>B</i>
eg B arphi ightarrow K eg B arphi	(Strong) negative introspection of B
$egin{array}{c} {\cal K}arphi ightarrow {\cal B}arphi \end{array}$	Knowledge implies Belief
Barphi o BKarphi	Full Belief
Inference Rules	
From φ and $\varphi \rightarrow \psi$ infer ψ .	Modus Ponens
From φ infer $K\varphi$.	Necessitation

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From φ infer $K\varphi$.	Necessitation

 Belief as subjective certainty: an agent "fully" believes φ iff she believes that she knows it.

Stalnaker's logic entails:

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Stalnaker's logic entails:



(Full) belief can be defined in terms of knowledge as

$$B\varphi \leftrightarrow \langle K \rangle K\varphi$$

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2 KD45 as the logic of belief

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- 2 KD45 as the logic of belief
- **3 S4.2** as the logic of knowledge

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Theorem (Folklore)

S4.2 is sound and complete wrt the class of extremally disconnected spaces (under the interior semantics).

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Theorem (Folklore)

S4.2 is sound and complete wrt the class of extremally disconnected spaces (under the interior semantics).

A space (X, τ) is called extremally disconnected if the closure of each open subset of X is open.

Topological Semantics for Full Belief

Our Proposal: Topological semantics for full belief

Topological Models for (Full) Belief and Belief Revision



Topological Semantics for Full Belief

Our Proposal: Topological semantics for full belief

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\mathsf{RECALL}: \vdash_{\mathsf{KB}} B\varphi \leftrightarrow \langle \mathsf{K} \rangle \mathsf{K} \varphi
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Topological Semantics for Full Belief

Our Proposal: Topological semantics for full belief

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\mathsf{RECALL}: \vdash_{\mathsf{KB}} B\varphi \leftrightarrow \langle \mathsf{K} \rangle \mathsf{K} \varphi
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Given an extremally disconnected space (X, τ) , we interpret belief as *the closure of the interior operator*:

 $\llbracket B\varphi \rrbracket = \operatorname{Cl}(\operatorname{Int}(\llbracket \varphi \rrbracket))$

Topological Models for (Full) Belief and Belief Revision

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The Most General Extensional Semantics for KB

- An extensional semantics for \mathcal{L}_{KB} : (X, K, B)
 - $\mathrm{K}:\mathcal{P}(X)\to\mathcal{P}(X)$
 - B : $\mathcal{P}(X) \to \mathcal{P}(X)$

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 - $\mathrm{K}:\mathcal{P}(X)\to\mathcal{P}(X)$
 - $\mathrm{B}:\mathcal{P}(X)\to\mathcal{P}(X)$
- An extensional model: (X, K, B, ν)

$$\llbracket \mathcal{K}\varphi \rrbracket^{\mathcal{M}} = \mathrm{K}\llbracket \varphi \rrbracket^{\mathcal{M}} \\ \llbracket \mathcal{B}\varphi \rrbracket^{\mathcal{M}} = \mathrm{B}\llbracket \varphi \rrbracket^{\mathcal{M}}$$

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$$\begin{bmatrix} K\varphi \end{bmatrix}^{\mathcal{M}} = K \llbracket \varphi \end{bmatrix}^{\mathcal{M}} \\ \begin{bmatrix} B\varphi \end{bmatrix}^{\mathcal{M}} = B \llbracket \varphi \end{bmatrix}^{\mathcal{M}} .$$

• A topological extensional semantics for \mathcal{L}_{KB} : $(X, K^{\tau}, B^{\tau}) := (X, Int, Cl(Int))$

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Topological Semantics for Full Belief

The Most General Extensional Semantics for KB

Theorem (Topological Representation Theorem)

An extensional semantics (X, K, B) validates all the axioms and rules of Stalnaker's system **KB** iff it is a topological extensional semantics given by an extremally disconnected topology τ on X.

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Future Work

Topological Semantics for Full Belief

Unimodal Cases: S4.2 and KD45

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Future Work

Topological Semantics for Full Belief

Unimodal Cases: S4.2 and KD45

Axioms of S4.2
$$K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)$$
 $K\varphi \rightarrow \varphi$ $K\varphi \rightarrow KK\varphi$ $\langle K \rangle K\varphi \rightarrow K \langle K \rangle \varphi$

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Stalnaker's Logic KB

Our Work

Future Work

Topological Semantics for Full Belief

Unimodal Cases: S4.2 and KD45

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Axioms of KD45
$m{B}(arphi ightarrow \psi) ightarrow (m{B} arphi ightarrow m{B} \psi)$
$m{B}arphi ightarrow \langle m{B} angle arphi$
$m{B}arphi ightarrow m{B}m{B}arphi$
$\langle \pmb{B} angle arphi ightarrow \pmb{B} \langle \pmb{B} angle arphi$

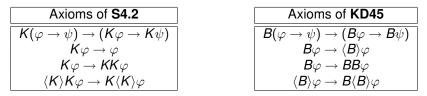
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Topological Semantics for Full Belief

Unimodal Cases: S4.2 and KD45



 Known: S4.2 is sound and complete wrt the class of extremally disconnected spaces.

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Topological Semantics for Full Belief

Unimodal Case: Completeness for KD45

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Topological Semantics for Full Belief

Unimodal Case: Completeness for KD45

$$(\mathcal{L}_{\mathcal{B}}) \varphi ::= \mathbf{p} \mid \neg \varphi \mid \varphi \land \varphi \mid \mathbf{B}\varphi$$

Theorem

KD45 is sound and complete wrt the class of extremally disconnected spaces.

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Static Belief Revision: Conditional Beliefs			

Conditional Beliefs

$(\mathcal{L}_{\mathsf{KCB}}) \ \varphi := p \mid \neg \varphi \mid \varphi \land \varphi \mid \mathsf{K}\varphi \mid \mathsf{B}\varphi \mid \mathsf{B}^{\psi}\varphi$

Topological Models for (Full) Belief and Belief Revision

Aybüke Özgün

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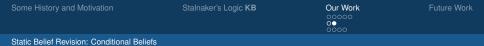
Some History and Motivation	Stalnaker's Logic KB	Our Work ○○○○○ ●○ ○○○○	Future Work
Static Belief Revision: Conditional Beliefs			

Conditional Beliefs

$$(\mathcal{L}_{\mathsf{KCB}}) \ \varphi := p \mid \neg \varphi \mid \varphi \land \varphi \mid \mathsf{K}\varphi \mid \mathsf{B}\varphi \mid \mathsf{B}^{\psi}\varphi$$

 B^ψφ := If the agent would learn ψ she would come to believe that φ was the case before the learning.

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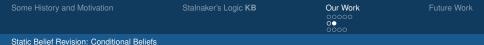
For any topological model *M* = (*X*, τ, ν) based on an extremally disconnected space (*X*, τ),

 $\llbracket B\varphi \rrbracket^{\mathcal{M}} = \operatorname{Cl}(\operatorname{Int}(\llbracket \varphi \rrbracket^{\mathcal{M}})) = \operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(\llbracket \varphi \rrbracket^{\mathcal{M}}))).$

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Topological Models for (Full) Belief and Belief Revision



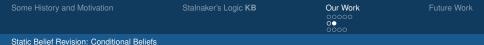
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• The Basic Topological Semantics for CB:

$$\llbracket B^{\varphi}\psi\rrbracket = \operatorname{Cl}(\llbracket \varphi\rrbracket^{\mathcal{M}} \cap \operatorname{Int}(\llbracket \varphi\rrbracket^{\mathcal{M}} \to \llbracket \psi\rrbracket^{\mathcal{M}})).$$

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For any topological model *M* = (*X*, τ, ν) based on an extremally disconnected space (*X*, τ),

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• The Basic Topological Semantics for CB:

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• The Refined Topological Semantics for CB:

$$\llbracket \boldsymbol{B}^{\varphi} \boldsymbol{\psi} \rrbracket = \operatorname{Int}(\llbracket \varphi \rrbracket^{\mathcal{M}} \to \operatorname{Cl}(\llbracket \varphi \rrbracket^{\mathcal{M}} \cap \operatorname{Int}(\llbracket \varphi \rrbracket^{\mathcal{M}} \to \llbracket \boldsymbol{\psi} \rrbracket^{\mathcal{M}}))).$$

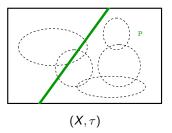
Some History and Motivation	Stalnaker's Logic KB	Our Work 00000 00 0000	Future Work
Dynamic Belief Revision: Updates			

Updates

Given a topological space (X, τ) and a set $P \subseteq X$, a space (P, τ_P) is called a subspace of (X, τ) where $\tau_P = \{U \cap P : U \in \tau\}.$

Updates

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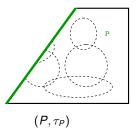


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Some History and Motivation	Stalnaker's Logic KB	Our Work ○○○○○ ○●○○	Future Work
Dynamic Belief Revision: Updates			

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Stalnaker's Logic KB

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Dynamic Belief Revision: Updates

 $\varphi := \boldsymbol{p} \mid \neg \varphi \mid \varphi \land \varphi \mid \boldsymbol{K} \varphi \mid \boldsymbol{B} \varphi \mid \boldsymbol{B}^{\psi} \mid \langle ! \varphi \rangle \varphi$

Topological Models for (Full) Belief and Belief Revision

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Some History and Motivation	Stalnaker's Logic KB	Our Work ○○○○○ ○○ ○○●○	Future Work
Dynamic Belief Bevision: Undates			

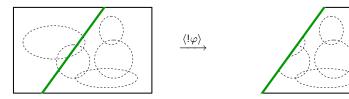
$$\varphi := \boldsymbol{\rho} \mid \neg \varphi \mid \varphi \land \varphi \mid \boldsymbol{K} \varphi \mid \boldsymbol{B} \varphi \mid \boldsymbol{B}^{\psi} \mid \langle ! \varphi \rangle \varphi$$

⟨!φ⟩ψ := φ is true and after the agent learns φ, ψ becomes true.

$$\varphi := p \mid \neg \varphi \mid \varphi \land \varphi \mid K\varphi \mid B\varphi \mid B^{\psi} \mid \langle !\varphi \rangle \varphi$$

⟨!φ⟩ψ := φ is true and after the agent learns φ, ψ becomes true.

$$\mathcal{M} = (X, \tau, \nu) \qquad \qquad \mathcal{M}_{\varphi} = (\llbracket \varphi \rrbracket, \tau_{\llbracket \varphi \rrbracket}, \nu_{\llbracket \varphi \rrbracket})$$



• $\llbracket \varphi \rrbracket = \llbracket \varphi \rrbracket^{\mathcal{M}}$

•
$$\tau_{\llbracket \varphi \rrbracket} = \{ U \cap \llbracket \varphi \rrbracket : U \in \tau \}$$

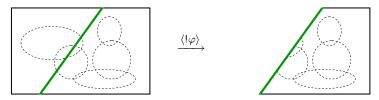
• $\nu_{\llbracket \varphi \rrbracket}(p) = \nu(p) \cap \llbracket \varphi \rrbracket$ for any $p \in \operatorname{Prop}$

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$$\varphi := \boldsymbol{\rho} \mid \neg \varphi \mid \varphi \land \varphi \mid \boldsymbol{K} \varphi \mid \boldsymbol{B} \varphi \mid \boldsymbol{B}^{\psi} \mid \langle ! \varphi \rangle \varphi$$

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$$\mathcal{M} = (X, \tau, \nu) \qquad \qquad \mathcal{M}_{\varphi} = (\llbracket \varphi \rrbracket, \tau_{\llbracket \varphi \rrbracket}, \nu_{\llbracket \varphi \rrbracket})$$



Given a topological model *M* = (*X*, *τ*, *ν*), the additional semantic clause reads

$$x \in \llbracket \langle ! \varphi \rangle \psi \rrbracket^{\mathcal{M}}$$
 iff $x \in \llbracket \varphi \rrbracket$ and $x \in \llbracket \psi \rrbracket^{\mathcal{M}_{\varphi}}$

Some History and Motivation	Stalnaker's Logic KB	Our Work 00000 00 0000	Future Work

Future Work

- Multi-agent case
- Action models
- Dynamic attitudes
- Relation with topo-logic and effort modality

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Thank you!

Topological Models for (Full) Belief and Belief Revision

Aybüke Özgün

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