Learning-theoretic take on: Belief Revision via Contraction

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Motivation

The role of fortune in science is apparent to everyone, and has led some to nihilism when questions arise when questions arise about justifying a method of inquiry (e.g., [Feyerabend 1975]).

The chaotic face of successful science, however, is due in part to the hazards of data collection and the mental limits of the scientists faced with the results.

In contrast, the idealized paradigms of inductive logic are free of such factors, so they make it easier to compare the merits of different scientific strategies, with a view to justifying one or more of them.

Martin, E., and Osherson, D. (1997). Scientific Discovery Based on Belief Revision, The Journal of Symbolic Logic, Vol. 62, No. 4, pp. 1352-1370.

Martin, E., and Osherson, D. (1998). Elements of Scientific Inquiry, Cambridge: MIT Press.

Scientific Strategy

Scientific strategy = a class of scientists

Definition

A strategy is canonical for a class C of problems just in case every solvable problem in C is solved by some scientist in this strategy.

Is a strategy reliable enough?

=

Is a class of scientists it canonical for a class C of (interesting) problems?

Outline

1. First-order Framework of Inquiry

2. Inquiry via Belief Revision

First-Order Logic

First-order framework uses first-order logic.

First-order Paradigm: Language I

To obtain the set of formulas $\mathcal{L}_{form}$, we fix:

Sym — a countable, decidable set of predicates and function symbols.

Var — a countably infinite set of variables.

First-order Paradigm: Language II

Further notation:

$Var = \{v_i \mid i \in \mathbb{N}\}$.

$\mathcal{L}_{sen} \subseteq \mathcal{L}_{form}$ — the set of sentences (no free variables).

$\mathcal{L}_{basic} \subseteq \mathcal{L}_{form}$ — the set of atomic formulas and the negations thereof.

First-order Paradigm: Structures

- Countable (finite or denumerable) structures.
- Structure $\mathcal{S}$ is a model of a set of formulas $\Gamma \subseteq \mathcal{L}_{form}$ iff there is an assignment $h : Var \to |\mathcal{S}|$, with $\mathcal{S} \models \Gamma[h]$.
- The class of models of $\Gamma \subseteq \mathcal{L}_{form}$ is denoted $MOD(\Gamma)$.

First-order Paradigm: Components

- Worlds.
- Problems.
- Environments.
- Scientists.
- Success.

WORLDS

All countable structures that interpret **Sym**.

Problems

A proposition is a non-empty class of structures.

A problem is a collection of disjoint propositions.

Example

Assume **Sym** contains only a single binary predicate. Let:
P_0 be a collection of strict total orders with a least point, and
P_1 be a collection of strict total orders without a least point.

Then $\mathbf{P} = \{P_0, P_1\}$ is a problem.

Environment

Sym is *observational*.

So is the domain: the elements are given temporary names.

Definition

Given structure $\mathcal{S}$, a full assignment to $\mathcal{S}$ is any mapping of Var onto $|\mathcal{S}|$.

Definition

Let structure $\mathcal{S}$ and a full assignment h to $\mathcal{S}$ be given.

1. An environment for $\mathcal{S}$ and h is a sequence e such that $range(e) = \{\beta \in \mathcal{L}_{basic} \mid \mathcal{S} \models \beta[h]\}$.
2. An environment for $\mathcal{S}$ is an environment for $\mathcal{S}$ and h, for some full assignment h to $\mathcal{S}$.
3. An environment is an environment for some structure.
4. An environment for proposition P is an environment for some $\mathcal{S} \in P$.
5. An environment for problem $\mathbf{P}$ is an environment for some $P \in \mathbf{P}$.

Environments: Examples

Suppose **Sym** $= \{R\}$, structure $|\mathcal{S}| = \mathbb{N}$, R is in fact $<$.

Example

h is a full assignment to $\mathcal{S}$ such that $\{(v_i, i) \mid i \in \mathbb{N}\}$. Then one environment for $\mathcal{S}$ and h looks like this:

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h is a full assignment to $\mathcal{S}$ such that $\{(v_i, i) \mid i \in \mathbb{N}\}$. Then one environment for $\mathcal{S}$ and h looks like this:

$$v_3 \neq v_4,\ \neg R v_0 v_0,\ R v_1 v_9,\ v_{11} = v_{11},\ v_0 \neq v_3, \ldots$$

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Example

g is a full assignment to $\mathcal{S}$ such that $\{(v_{2i}, i), (v_{2i+1}, i) \mid i \in \mathbb{N}\}$. Then one environment for $\mathcal{S}$ and h looks like this:

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Example

g is a full assignment to $\mathcal{S}$ such that $\{(v_{2i}, i), (v_{2i+1}, i) \mid i \in \mathbb{N}\}$. Then one environment for $\mathcal{S}$ and h looks like this:

$$v_2{=}v_3,\ \neg Rv_4v_5,\ Rv_1v_9,\ v_{11}{=}v_{11},\ v_0{\neq}v_3, \ldots$$

Environments and Structure Isomorphism

Lemma

Let two structures $\mathcal{S}$ and $\mathcal{T}$ be given.

1. *if $\mathcal{S}$ and $\mathcal{T}$ are isomorphic then the set of environments for $\mathcal{S}$ is identical to the set of environments for $\mathcal{T}$.*
2. *if some environment is both for $\mathcal{S}$ and $\mathcal{T}$ then $\mathcal{S}$ and $\mathcal{T}$ are isomorphic.*

Environments: Notation

- Take environment e and $k \in \mathbb{N}$. Then:
  - e_k is k-th element of e, and
  - $e[k]$ is the initial segment of e of length k.
- SEQ denotes the collection of proper initial segments of any environment.
- Let $\sigma \in SEQ$, if σ is non-void, then $\bigwedge \sigma$ is the conjunction of the formulas in $range(\sigma)$; if σ is void, then $\bigwedge \sigma$ is $\forall v_0(v_0 = v_0)$.
- $Var(\sigma)$ is the set of all free variables in σ.
- Given a proposition P and $\sigma \in SEQ$, we say that σ is for P just in case $\bigwedge \sigma$ is satisfiable in some member of P (similarly for $\mathbf{P}$).

Scientists

A scientist Ψ is a partial or total mapping from SEQ into classes of structures.

If scientist Ψ is defined on $\sigma \in SEQ$, then $\Psi(\sigma)$ is a collection of structures, thus a proposition.

Success

Definition

Let scientist Ψ be given.

1. Let environment e for proposition P be given. We say that Ψ solves P in e just in case for cofinitely many k, $\emptyset \neq \Psi(e[k]) \subseteq P$. We say that Ψ solves P just in case Ψ solves P in every environment for P.
2. Let problem $\mathbf{P}$ be given. We say that Ψ solves $\mathbf{P}$ just in case Ψ solves every member of $\mathbf{P}$. In this case we say that $\mathbf{P}$ is solvable.

Solvability: Examples

Example

Sym={H}, where H is a unary predicate. Given $n \in \mathbb{N}$, let P_n be the class of all structures $\mathcal{S}$ such that $card(H^{\mathcal{S}}) = n$.
$\mathbf{P} = \{P_n \mid n \in \mathbb{N}\}$ is

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Example

Sym$=\{$R$\}$, where R is a binary predicate. Set
$P_y = \{\langle \mathbb{N}, \prec \rangle \mid \prec$ is isomorphic to $\omega\}$,
$P_n = \{\langle \mathbb{N}, \prec \rangle \mid \prec$ is isomorphic to $\omega^*\}$.
$\mathbf{P} = \{P_y, P_n\}$ is

Solvability: Examples

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First step towards characterization: locking pairs

Definition

Let scientist Ψ, proposition P, $\mathcal{S} \in P$, $\sigma \in SEQ$, and finite assignment $a : Var \rightarrow |\mathcal{S}|$ be given. We say that (σ, a) is a locking pair for Ψ, $\mathcal{S}$ and P just in case the following conditions hold.

1. $Var(\sigma) \subseteq domain(a)$
2. $\mathcal{S} \models \bigwedge \sigma[a]$
3. For every $\tau \in SEQ$, if $\mathcal{S} \models \exists \bar{x} \bigwedge (\sigma * \tau)[a]$, where $\bar{x}$ contains the variables in $Var(\tau) - domain(a)$, then $\emptyset \neq \Psi(\sigma * \tau) \subseteq P$.

Lemma

Take a scientist Ψ, proposition P, and $\mathcal{S} \in P$. Suppose that scientist Ψ solves P in every environment for $\mathcal{S}$. Then there is a locking pair for Ψ, $\mathcal{S}$, and P.

Characterization: tip-offs

Definition

A $\pi - set$ is any collection of $\forall$ formulas all of whose free variables are drawn from the same finite set.

Definition

Let problem $\mathbf{P}$ and $P \in \mathbf{P}$ be given. A tip-off for $P \in \mathbf{P}$ is a countable collection $\mathbf{t}$ of π-sets such that:

1. for every $\mathcal{S} \in P$ and full assignment h to $\mathcal{S}$, there is $\pi \in \mathbf{t}$ with $\mathcal{S} \models \pi[h]$;
2. for all $\mathcal{U} \in P' \in \mathbf{P}$ with $P' \neq P$, all full assignments g to $\mathcal{U}$, and all $\pi \in \mathbf{t}$, $\mathcal{U} \not\models \pi[g]$.

If every member of $\mathbf{P}$ has a tip-off in $\mathbf{P}$, then we say that $\mathbf{P}$ has tip-offs.

Characterization

Proposition

If problem **P** *is countable and has tip-offs, then* **P** *is solvable.*

Proposition

Every solvable problem has tip-offs.

Outline

Introduction

- Inquiry within the first-order paradigm
- as a process of 'rational' belief revision
- in the light of data,
- starting from a background theory.

The scientist starts her inquiry with a set of formulas X.

The idea here is similar as in AGM.

But we start off with $X \subset \mathcal{L}_{form}$ as the state of belief, without assuming X to be deductively closed.

Contraction

Definition

Let $\varphi \in \mathcal{L}_{form}$ and $B \subseteq \mathcal{L}_{form}$ be given. By a maximal subset of B that fails to imply φ is meant any subset B' of B with the following properties:

1. $B' \not\models \varphi$;
2. there is no X with $B' \subset X \subseteq B$ and $X \not\models \varphi$.

The class of all maximal subsets of B that fail to imply φ is denoted by $B \perp \varphi$.

Lemma

$\bigcap(B \perp \varphi) = \{\psi \in B \mid \textit{for all } D \subseteq B \textit{ if } D \cup \{\psi\} \models \varphi \textit{ then } D \models \varphi\}$.

Definition

A mapping $\dot{-}$ from $\mathcal{P}(\mathcal{L}_{form}) \times \mathcal{L}_{form}$ to $\mathcal{P}(\mathcal{L}_{form})$ is a contraction function just in case for all $B \subseteq \mathcal{L}_{form}$ and $\varphi \in \mathcal{L}_{form}$:

1. $\bigcap(B \perp \varphi) \subseteq B \dot{-} \varphi \subseteq B$;
2. if $\not\models \varphi$ then $B \dot{-} \varphi \not\models \varphi$.

Special Kinds of Contraction

Definition

A contraction function $\dot{-}$ is maxichoice just in case of every $B \subseteq \mathcal{L}_{form}$ and invalid $\varphi \in \mathcal{L}_{form}$, $B \dot{-} \varphi \in B \perp \varphi$.

Definition

A contraction function $\dot{-}$ is stringent just in case there is a strict total ordering $\prec$ of $\mathcal{P}(\mathcal{L}_{form})$ such that for all $B \subseteq \mathcal{L}_{form}$ and invalid $\varphi \in \mathcal{L}_{form}$, $B \dot{-} \varphi$ is the $\prec$-least subset of B that does not imply φ.

Proposition

Every stringent contraction function is maxichoice.

Revision Defined from Contraction

Definition

A mapping $\dot{+}$ from $\mathcal{P}(\mathcal{L}_{form}) \times SEQ$ to $\mathcal{P}(\mathcal{L}_{form})$ is revision function just in case there is a contraction function $\dot{-}$ such that for all $B \subseteq \mathcal{L}_{form}$ and $\sigma \in SEQ$,

$$B \dot{+} \sigma = \begin{cases} B & \text{if } \sigma = \emptyset \\ (B \dot{-} \neg \bigwedge \sigma) \cup range(\sigma) & otherwise \end{cases}$$

Revision Defined from Contraction

Lemma

Let revision function $\dot{+}$ be given. Then for all $B \subseteq \mathcal{L}_{form}$ and $\sigma \in SEQ$:

1. $B \dot{+} \sigma \models \bigwedge \sigma$.
2. $B \dot{+} \sigma \subseteq B \cup range(\sigma)$.
3. *If* $B \not\models \neg \bigwedge \sigma$ *then* $B \dot{+} \sigma = B \cup range(\sigma)$.
4. *If* σ *is non void then* $B \dot{+} \sigma$ *is consistent.*

Inquiry via Revision

$$\lambda\sigma \,.\, B \dot{+} \sigma$$

Linguistic Scientists

Definition

Scientist Ψ is linguistic just in case there is $\psi : SEQ \rightarrow \mathcal{P}(\mathcal{L}_{form})$ such that for all $\sigma \in SEQ$, $\Psi(\sigma)$ is defined iff $\psi(\sigma)$ is defined, and when both are defined $\Psi(\sigma) = MOD(\psi(\sigma))$. In this case, we say that ψ underlies Ψ.

Scientists Based on Revision

Definition

Let revision function $\dotplus$ be given. Then $\lambda\sigma.B\dotplus\sigma$ is a linguistic scientist, which we qualify as revision-based.

The Inductive Power of Stringent Revision

Theorem

There is a stringent revision function $\dot{+}$ with the following property. Let problem **P** *be such that for some $Y \subseteq \mathcal{L}_{form}$ and revision function $\dot{\oplus}$, $\lambda\sigma.Y\dot{\oplus}\sigma$ solves* **P**. *Then there is a consistent $X \subseteq \mathcal{L}_{form}$ such that $\lambda\sigma.Y\dot{+}\sigma$ solves* **P**.

The Inductive Power of Stringent Revision

Theorem

There is a stringent revision function $\dot{+}$ and propositions $\{P_1, P_2\}$ such that:

- $\{P_1, P_2\}$ *is solvable;*
- *for all* $B \subseteq \mathcal{L}_{form}$, $\lambda\sigma B \dot{+} \sigma$ *does not solve* $\{P_1, P_2\}$.

END OF DAY 3