Introduction: Formal Learning Theory

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True, there are good reasons for preferring the computable way of deriving knowledge. We know the results of computations, and only think we know the results of trial and error procedures [viz. limiting computation]. There are many reasons for preferring knowing to thinking (as Popper observed). But that does not change the fact that sometimes thinking may be more appropriate.

Kugel, Thinking may be more than computing

- A class of possible worlds (known by both players).
- Nature chooses one of them (Scientist does not know which).
- Nature generates data about the world.
- On the basis of this inductively given data Scientist draws his conjectures.
- Each time a new information comes in, Scientist can answer with a different hypothesis.
- Scientist succeeds if the sequence of his answers stabilizes to a correct hypothesis.

Whether Scientist succeeds or not, depends on her skills and on the problem.

1. Possible realities.
2. A scientific problem.
3. For each reality, a set of data streams of accessible information.
4. Scientist (Learner).
5. Success criterion.

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Strict total orders $\prec$ over $\mathbb{N}$, e.g.:

A) 0 1 2 3 4 5 6 7 . . .

B) 2 1 0 5 4 3 8 7 . . .

C) . . . 7 6 5 4 3 2 1 0

D) 0 2 4 5 6 . . . 1 3 5 7 9 . . .

E) . . . 11 9 7 5 3 1 0 2 4 6 8 . . .

F) 0 . . . 11 9 7 5 3 2 4 6 8 10 . . . 1

An example: Orders

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A problem is a collection of orders, e.g.:

A) 0 1 2 3 4 5 6 7 . . .

B) 2 1 0 5 4 3 8 7 . . .

C)

D) 0 2 4 5 6 . . . 1 3 5 7 9 . . .

E)

F) 0 . . . 11 9 7 5 3 2 4 6 8 10 . . . 1

An example: Orders

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An environment for a given $\prec$ is any enumeration of all facts $i \prec j$ (R(i,j)), e.g:

A) R(2,3) R(1,2) R(0,2) R(1,4) R(0,3) R(0,4) ...

B) R(5,10) R(0,4) R(11,1) R(0,2) R(11,9) R(0,6) ...

Definition

e is an environment for order $\prec$ iff the elements of e form the set:

$$\{R(i,j) \mid i,j \in \mathbb{N} \text{ and } i \prec j\}.$$

Repetitions allowed

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Lemma

Let e be for two orders $\prec_1$ and $\prec_2$. Then $\prec_1$ and $\prec_2$ are identical.

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Orders: Scientists

Here, the scientist wants to know whether the order has a least element.

- She receives data from an environment in a piecemeal fashion
- hence she receives finite sequences of environments (elements from *SEQ*)
- and outputs 'yes' or 'no' (i.e., elements from $\{0, 1\}$).

Example

$\emptyset$

$\langle R(2,3)\rangle$

$\langle R(2,3), R(1,2)\rangle$

$\langle R(2,3), R(1,2), R(0,2)\rangle$

$\langle R(2,3), R(1,2), R(0,2), R(1,4)\rangle$

$\vdots$

$\langle R(2,3), R(1,2), R(0,2), R(1,4), R(0,3), R(0,4), R(0,5), R(1,3)\rangle$

The Scientist might answer with *yes* on first two then with *no*, etc.

Note that *SEQ* is countable.

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Take e, for $k \in \mathbb{N}$, $e|k$ denotes the finite initial segment of e of length k, e.g.:

- $\emptyset, \langle R(2,3)\rangle, \langle R(2,3), R(1,2)\rangle, \langle R(2,3), R(1,2), R(0,2)\rangle$
- $e[0], e[1], e[2], e[3]$

  This produces a sequence of hypotheses of the scientist:
- $\Psi(e[0]), \Psi(e[1]), \Psi(e[2]), \Psi(e[3])$, e.g.,
- 1, 1, 0, 1

Definition

We say that Ψ solves problem **P** just in case for every $\prec \in$ **P** and every environment e for $\prec$, the following conditions hold:

1. If $\prec$ has a least element then for co-finitely many k, $\Psi(e[k]) = 1$.
2. If $\prec$ has no least element then for co-finitely many k, $\Psi(e[k]) = 0$.

If some scientist solves **P** then **P** is solvable, otherwise it is unsolvable.

1) Let **P** consist of every order that is isomorphic either to ω or ω^*.
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- N. Chomsky (1957).
- H. Putnam, R.J. Solomonoff and E. M. Gold ('60).
- Theory of scientific/empirical inquiry, formal learning theory, machine learning, computational learning theory, grammar inference.

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Any language, i.e., a set of natural numbers.

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Any collection of languages.

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Definition

An environment e is any sequence over $\mathbb{N}$.
e is for L just in case $range(e) = L$.
e is for $\mathbf{P}$ just in case e is for some $L \in \mathbf{P}$.

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Any function Ψ from *SEQ* to $\mathcal{P}(\mathbb{N})$.

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Definition

Let Ψ, e, $\mathbf{P}$ and $L \in \mathbf{P}$ be given. Then:

1. Ψ converges on e to L iff for all but finitely many $k \in \mathbb{N}$, $\Psi(e[k]) = L$.
2. Ψ identifies L in the limit iff Ψ converges to L on every e for L.
3. Ψ identifies $\mathbf{P}$ in the limit iff Ψ identifies in the limit every $L \in \mathbf{P}$.

Lemma

Every identifiable in the limit collection of languages in countable.

Example

- Let $\mathcal{C} = \{\mathbb{N} - \{x\} \,|x \in \mathbb{N}\}$.
- E.g. $\{0, 1, 3, 4, 5, 6 \ldots\} \in \mathcal{C}$
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FACT

For all sets from $\mathcal{C}$ the above guessing rule is a winning strategy for Ψ.

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Fact

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If we require that the incoming information is arranged increasingly $\rightarrow$ there is a winning strategy.

Definition

Let Ψ, $L \in \mathbf{P}$, and $\sigma \in SEQ$ be given. σ is a locking sequence for Ψ and L iff:

1. $\Psi(\sigma)$ is defined.
2. for all $\tau \in SEQ$ of elements of L and extending σ, $\Psi(\tau) = \Psi(\sigma)$.

In other words, σ locks Ψ on the hypothesis $\Psi(\sigma)$ — new data from L can not lead to a change of Ψ's mind.

Lemma (Blum&Blum 1975)

Let Ψ, $L \in \mathbf{P}$ be given, such that Ψ identifies L in the limit. Then there is a locking sequence σ for Ψ and L. Moreover, $\Psi(\sigma) = L$.

Fact

The class **F**, *that includes all finite sets is identifiable in the limit.*

Argument: For all $\sigma \in SEQ$, $L(\sigma)$ is the least integer from $set(\sigma)$.

No extension of F is identifiable in the limit.

Theorem (Anguin 1980)

Let $\mathbf{P}$ *be a class of sets.* $\mathbf{P}$ *is identifiable in the limit iff for all* $L \in \mathbf{P}$ *there is a finite set* $D_L \subseteq L$ *such that there is no* $L' \in \mathbf{P}$*, if* $D_L \subseteq L' \subset L$*.*

EXAMPLE

- Let $C_0 = \{\{0,1\},\{0,2\},\{0,3\},\ldots\}$. .
- Let $C_1 = \{\{1\},\{1,2\},\{1,2,3\}\}$.

Definition

Let Ψ, $\mathbf{P}$ and $L \in \mathbf{P}$ and e for L be given. Then:

1. Ψ finitely identifies L on e iff there is $k \in \mathbb{N}$, such that $\Psi(e[k]) = L$, and Ψ stops.
2. Ψ finitely identifies L iff Ψ finitely identifies L on every e for L.
3. Ψ finitely identifies $\mathbf{P}$ iff Ψ finitely identifies every $L \in \mathbf{P}$.

Definition

We call any $L \subseteq \mathbb{N}$ a language. An indexed family of recursive languages is a class $C = \{L_0, L_1, \ldots\}$ for which a computable function $f : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ exists that uniformly decides C, i.e.,

$$f(i, w) = \begin{cases} 1 & \text{if } w \in L_i, \\ 0 & \text{if } w \notin L_i. \end{cases}$$

.

- Learner is a computable function.
- FTTs are computably generated.

1. Possible realities: functional languages.

   $L \subseteq \mathbb{N}$ is functional if L gives a graph of a (total) function from $\mathbb{N}$ to $\mathbb{N}$
2. A scientific problem: class of functional languages.
3. Data streams.
4. Scientist (Learner): the same as before.
5. Success criterion: the same as before.

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Fact

Let e be the environment and suppose that both range(e) and L are functional languages. Then, for all $(x, y) \in L$, if $(x, y) \notin$ range(e), then $(x, y') \in$ range(e) for some $y' \neq y$.

1. Data and conclusions are of a different nature.
2. Inductive, step-by-step process.
3. Starts with a class of hypotheses.
4. Potentially infinite procedure, defined in the limit.
5. Results in operational, non-introspective knowledge.
6. Single-agent learning.
7. Environments — only true, positive and readable info.

Data and conclusions are of a different nature

1. Learning that φ — epistemic one-step update, adding φ.
2. Here the incoming information is spread over more steps.
3. Different nature of incoming data and conclusions.
4. At each stage only partial information about a set.
5. Sentences and grammars, natural numbers and TM.
6. Knowing the hypothesis $\rightarrow$ knowing what data.
7. No conclusive inference from data to hypotheses.

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1. In LT the convergence point is unknown and uncomputable.
2. Finite sequences are not single announcements of hypothesis.

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Learning starts with a class of hypotheses

1. Learning starts with a class of hypotheses.
2. It represents the background knowledge of Scientist.
3. So, Scientist expects that one of them is true, and he is right about it.
4. Picking *one element from a class*!

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Learning is a potentially infinite procedure, defined in the limit

1. LT defined for potentially infinite universes.
2. Even for finite worlds environments are infinite.
3. Scientist does not know the finiteness or size of the entity.
4. Scientist can never know if all the elements have already been enumerated.
5. This leads to infinite procedures and conditions defined in the limit.

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Learning results in operational, non-introspective knowledge

1. Learning leads to kind-of knowledge. Scientist:
   - declares a hypothesis that is true;
   - believes that it is true;
   - has a justification to choose it.
2. Strictly operational nature.
3. Not introspective.
4. No way to point out the right guess, might be forced to change his mind.

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1. One agents?
2. Two agents?
3. More?
4. Usually, in LT only one player. Nature is objective.

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Environments include only true, positive and readable information

1. Truthfulness.
2. Positiveness.
3. Readability.

END OF DAY 2